PHYSICAL REVIEW E VOLUME 50, NUMBER 2 AUGUST 1994

Near-equilibrium multiple-wave plasma states

Mark Buchanan
Department of Physics, University of Virginia, Charlottesville, Virginia 22903-2442

J. Dorning
Engineering Physics Program and Department of Applied Mathematics, University of Virginia, Charlottesville, Virginia 22903-2442

(Received 4 April 1994)

We report results showing that spatially periodic Bernstein-Greene-Kruskal (BGK) waves, which are exact nonlinear traveling wave solutions of the Vlasov-Maxwell equations for collisionless plasmas, satisfy a nonlinear principle of superposition in the small-amplitude limit. For an electric potential consisting of N traveling waves, $\varphi(x,t)=\sum_{i=1}^{N}\varphi^{(i)}(x-v_i t)$, where v_i is the velocity of the ith wave and each wave amplitude $\varphi^{(i)}$ is of order ϵ which is small, we first derive a set of quantities $\overline{\mathcal{E}}^{(i)}(x,u,t)$ which are invariants through first order in ϵ for charged particle motion in this N-wave field. We then use these functions $\overline{\mathcal{E}}^{(i)}(x,u,t)$ to construct smooth distribution functions for a multispecies plasma which satisfy the Vlasov equation through first order in ϵ uniformly over the entire x-u phase plane for all time. By integrating these distribution functions to obtain the charge and current densities, we also demonstrate that the Poisson and Ampère equations are satisfied to within errors that are $O(\epsilon^{3/2})$. Thus the constructed distribution functions and corresponding field describe a self-consistent superimposed N-wave solution that is accurate through first order in ϵ. The entire analysis explicates the notion of small-amplitude multiple-wave BGK states which, as recent numerical calculations suggest, is crucial in the proper description of the time-asymptotic state of a plasma in which a large-amplitude electrostatic wave undergoes nonlinear Landau damping.

PACS number(s): 52.35.Fp, 52.35.Mw, 52.35.Sb

I. INTRODUCTION

The nonlinear Vlasov-Maxwell model of kinetic theory [1] presupposes that a sufficiently rarefied plasma can be adequately described by a set of smoothly varying distribution functions $f_\alpha(\mathbf{x},\mathbf{u},t)$, $\alpha=1,2,\ldots,M$, which characterize the distribution of each of M species over the single-particle $(\mathbf{x},\mathbf{u})$ phase space. By coupling the f_α to Maxwell's equations, one arrives at a self-consistent model for the plasma dynamics;

$$\nabla\cdot\mathbf{E}=4\pi\rho(\mathbf{x},t)=4\pi\sum_\alpha q_\alpha\int d^3u\, f_\alpha\ , \tag{1}$$

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0}\ , \tag{2}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{3}$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{j}(\mathbf{x},t)=\frac{4\pi}{c}\sum_\alpha q_\alpha\int d^3u\,\mathbf{u}f_\alpha\ , \tag{4}$$

$$\frac{\partial f_\alpha}{\partial t}+\mathbf{u}\cdot\nabla f_\alpha+\frac{q_\alpha}{m_\alpha}\mathbf{E}\cdot\nabla_{\mathbf{u}}f_\alpha=0\ , \tag{5}$$

where Eqs. (1)–(4) are Maxwell's equations, $\rho(\mathbf{x},t)$ and $\mathbf{j}(\mathbf{x},t)$ being the charge and current densities, and Eq. (5), the Vlasov or collisionless Boltzmann equation, describes the local conservation of particles in the $(\mathbf{x},\mathbf{u})$ phase space. This model is particularly appropriate for plasmas in the collisionless regime, wherein the effects of particle-particle interactions are dominated by collective plasma dynamics. Since any set of distribution functions $F_\alpha(\mathbf{u})$ which yields vanishing charge and current densities satisfies Eqs. (1)–(5), there exists an infinite manifold of solutions describing spatially uniform field-free collisionless plasma equilibria, the so-called "Vlasov equilibria." In an actual physical system the states corresponding to these equilibria in fact represent "metaequilibria" which should, given enough time, evolve toward thermal equilibrium. But often the time scale over which thermal equilibrium is attained is very long compared to that over which collective plasma processes occur. Thus, on the time scales relevant to these collective phenomena, Eqs. (1)–(5) appropriately treat the Vlasov equilibria as true stationary states of the plasma.

Simplified models appropriate for particular cases of physical interest can be derived from the full set of Eqs. (1)–(5). One of the most frequently employed models describes a plasma embedded in a strong, homogeneous magnetic field where the macroscopic particle motion is constrained to take place predominantly along the direction of the field. For nonrelativistic processes in such plasmas the collective dynamics are primarily electrostatic, i.e., are driven by the nonuniform distribution of charge, and Eqs. (1)–(5) can be reduced to one-dimensional nonlinear Vlasov-Poisson-Ampère equations;

$$\frac{\partial f_\alpha}{\partial t}+u\frac{\partial f_\alpha}{\partial x}+\frac{q_\alpha}{m_\alpha}E\frac{\partial f_\alpha}{\partial u}=0\ , \tag{6}$$

$$\frac{\partial E}{\partial x}=4\pi\sum_\alpha q_\alpha\int_{-\infty}^{\infty}du\, f_\alpha\ , \tag{7}$$

1063-651X/94/50(2)/1465(14)/$06.00

$$-\frac{\partial E}{\partial t}=4\pi\sum_{\alpha}q_\alpha\int_{-\infty}^{\infty}du\, u f_\alpha , \qquad (8)$$

where $E(x,t)$ is the self-consistent longitudinal electric field (the component of **E** along the ambient **B** field). The distribution function f_α in Eqs. (6)–(8) is obtained from that in Eqs. (1)–(5) by integration over the components of velocity transverse to **B**.

The theory of longitudinal plasma processes developed on the basis of Eqs. (6)–(8) has been applied broadly to the study of plasma phenomena in diverse settings. Since these equations are inherently nonlinear, however, much of our understanding of wave processes in near-equilibrium plasmas, including the well-known phenomenon of collisionless damping of small-amplitude waves in a broad class of plasmas, follows from Landau's classic analysis [2] of the initial value problem for Eqs. (6)–(8) linearized about a charge and current neutral equilibrium $F_\alpha(u)$. Landau arrived at an expression for the time-asymptotic value of the kth spatial Fourier mode of the electric potential,

$$\varphi_k(t)\underset{t\to\infty}{\sim}c_k(\bar{\lambda}_k)e^{\bar{\lambda}_k t} , \qquad (9)$$

where $c_k(\bar{\lambda}_k)$ depends upon the initial perturbation of the plasma, and $\bar{\lambda}_k$ is that root of the Landau dispersion relation

$$D_F(k,\lambda)=1-\frac{4\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_L du\frac{F'_\alpha(u)}{u-i\lambda/k}=0 \qquad (10)$$

with the least negative real part. (In the complex u plane, the Landau contour L runs from $-\infty$ to $+\infty$ along the real u axis, but is deformed to pass under the pole at $u=i\lambda/k$.) The time-asymptotic behavior of Eq. (9) depends solely upon the distribution functions $f_\alpha(u)$ of the Vlasov equilibrium, since these determine, through the function $D_F(k,\lambda)$, whether $\mathrm{Re}(\bar{\lambda}_k)$ is positive or negative. Importantly, the linear analysis predicts the time-asymptotic exponential damping $[\mathrm{Re}(\bar{\lambda}_k)<0]$ of all sufficiently small and smooth perturbations of so-called "single-humped" equilibria, for which $\sum_\alpha(q_\alpha^2/m_\alpha)F_\alpha(u)$ has a single maximum as a function of u [3]. One example of great practical importance is the thermal equilibrium plasma in which each species has a Maxwellian velocity distribution

$$F_\alpha(u)=n_\alpha(m_\alpha/2\pi kT_\alpha)^{3/2}\exp(-m_\alpha u^2/kT_\alpha) .$$

Experiments [4,5] have substantiated the reality of Landau damping in laboratory plasmas; moreover, Dawson [6] has given a simple physical explanation of the phenomenon in terms of the resonant transfer of energy between the wave and the particles of the plasma. In general, insights gained from the analysis of the linearized model have contributed greatly to the understanding of plasma dynamics in a wide variety of settings ranging from laboratory plasmas in thermonuclear fusion devices [7] and particle accelerating machines, for instance, to naturally occurring near-earth, interplanetary, solar, and astrophysical plasmas [8].

Analysis of the full nonlinear Vlasov-Poisson-Ampère system [Eqs. (6)–(8)] is extremely difficult, not only because of nonlinearity, but also because of the existence of the continuous manifold of equilibria. Consider the problem of determining the subsequent evolution of a plasma that is perturbed slightly from one of these equilibrium states. Since the equilibria form a continuous manifold, in any neighborhood no matter how small of a particular Vlasov equilibrium $F_\alpha(u)$, there always exist other equilibria. Thus it is always possible, even for a perturbation that is arbitrarily small, that the state of the plasma after the perturbation will in fact be closer to some other equilibrium in the manifold than it is to the original. This suggests, at the very least, that any analysis based upon linearization about the original equilibrium, which attempts to determine the plasma dynamics solely from the properties of that equilibrium, is going to be inadequate.

In fact, it has been demonstrated conclusively that neither Landau's linearized analysis, nor any of the other prominent and physically equivalent linear analyses [9,10], tells the complete story even of small-amplitude plasma waves. That nonlinear traveling waves of small *but constant amplitude* can exist in many plasmas was first recognized by Bohm and Gross [11] as long ago as 1949. Such waves, which do not exhibit Landau damping, are distinguished by the existence of plasma particles that are trapped within the electrostatic potential wells formed by the wave. This particle trapping phenomenon, which is ignored by traditional linear theories, opposes wave damping through modification of the space-averaged plasma distribution functions. By formalizing the methods of Bohm and Gross, Bernstein, Greene, and Kruskal [12] in 1957 characterized a class of basic exact undamped nonlinear solutions of the Vlasov-Maxwell equations, which have become known variously as BGK modes, BGK waves, or BGK equilibria. These authors showed that it is possible to choose the distributions of such trapped particles appropriately so as to create electrostatic plasma waves with an essentially arbitrary relationship between frequency ω and wave number k. In recent years Holloway and Dorning [13] have provided details for a rigorous theory of these waves in the small-amplitude limit, and in particular have given a precise nonlinear description of the particle distribution functions. For BGK waves of small amplitude, it turns out that the relationship between ω and k is less arbitrary, and is given in the limit of zero amplitude by the Vlasov dispersion relation [1,11,13]

$$1-(4\pi/k^2)\sum_\alpha(q_\alpha^2/m_\alpha)\mathrm{P}\int du\frac{F'_\alpha(u)}{u-\omega/k}=0 , \qquad (11)$$

where P denotes the principle value, and the functions $F_\alpha(u)$ describe the equilibrium plasma state in the absence of waves. Practical methods have recently been developed [14–16] and used to characterize the full spectrum of such undamped waves that can exist nearby physically relevant plasma equilibria. To develop the background for the rest of this paper, we shall briefly review the theory of small amplitude BGK waves in Sec. II.

In this paper we shall consider interactions which occur between a discrete set of spatially periodic small-amplitude BGK waves, in the case when the waves have appreciably different velocities. The analysis, which focuses primarily on the interaction of two spatially periodic small-amplitude BGK waves, is motivated by a desire to explain the results of recent large scale numerical simulations of the one-dimensional Vlasov-Maxwell equations. Due to the essentially nonlinear phenomenon of particle trapping, which remains important at any wave amplitude, a linear superposition of two spatially periodic BGK waves does not yield a solution, even in the limit of zero wave amplitude. We shall demonstrate this fact explicitly in Sec. III. Nevertheless, recent numerical simulations of the Vlasov-Poisson system performed by Demeio and Zweifel [17] strongly suggest that superpositions of such waves are necessary to the proper description of the time-asymptotic state obtained when a large-amplitude electrostatic wave suffers nonlinear Landau damping. Superimposed wave states also appear time asymptotically in the evolution of the two-stream and bump-on-tail instabilities. In each of these three cases, numerical studies indicate that the final state electric fields are, to a very good approximation, simple linear superpositions of two traveling electrostatic waves, which suggests the possible existence of a nonlinear superposition principle for small-amplitude spatially periodic BGK waves. The main goal of this paper is to develop this nonlinear superposition principle through the explicit construction of the distribution functions of a self-consistent superimposed two-wave state.

Central to this nonlinear superposition problem is the nonintegrable Hamiltonian system that describes the motion of a single charged particle in the field of two spatially periodic electrostatic waves. We shall consider this dynamical problem in Sec. IV, and obtain a clear qualitative picture of the phase flow in the single-particle phase space. To describe this flow quantitatively, at least approximately, we use perturbation methods to develop two first-order invariants, generalizations of the single-particle energy, which capture the gross features of the dynamics, including the primary resonance regions corresponding to each of the two waves. After developing these invariants in Sec. IV, we then exploit them in Sec. V to construct smooth distribution functions for a two-wave state. These functions satisfy the Vlasov equation uniformly through first order in the wave amplitudes and also generate, through the charge and current densities which enter into the inhomogeneous Maxwell equations, the correct self-consistent field. The result, which we generalize to the case of N waves in Sec. VI, shows that small-amplitude periodic BGK waves of sufficiently different velocity do satisfy a nonlinear superposition principle in which the fields superimpose linearly while the distribution functions combine according to a more complicated but well-defined rule. In Sec. VII we shall review the large scale numerical simulations of Demeio and Zweifel [17] and discuss the relevance of the superposition principle developed here to the description of the asymptotic plasma states these authors observed.

II. SMALL-AMPLITUDE BGK WAVES

In this section we shall briefly summarize the theory of spatially periodic small-amplitude BGK waves as it is developed in Refs. [13–16]. In searching for single traveling wave solutions to Eqs. (6)–(8) of the form

$$f_\alpha(x,u,t)=f_\alpha(x-vt,u-v)\ , \tag{12}$$

$$E(x,t)=E(x-vt)\ , \tag{13}$$

we shall be particularly interested in solutions that are "nearby" a Vlasov equilibrium with smooth distribution functions $F_\alpha(u)$, i.e., solutions for which both $h_\alpha=f_\alpha-F_\alpha$ and E are small everywhere. It is convenient for analysis to transform Eqs. (6)–(8) to the wave frame with variables $\xi=x-vt$ and $w=u-v$. Then, upon introducing the electric potential φ, and writing $f_a(\xi,w)$ in terms of its even and odd parts with respect to w [13],

$$f^e_\alpha(\xi,w)=\tfrac{1}{2}[f_\alpha(\xi,w)+f_\alpha(\xi,-w)]\ , \tag{14}$$

$$f^o_\alpha(\xi,w)=\tfrac{1}{2}[f_\alpha(\xi,w)-f_\alpha(\xi,-w)]\ , \tag{15}$$

we find that Eqs. (6)–(8) separate into two independent pairs of equations:

$$w\frac{\partial f^e_\alpha}{\partial\xi}-\frac{q_\alpha}{m_\alpha}\frac{d\varphi}{d\xi}\frac{\partial f^e_\alpha}{\partial w}=0\ , \tag{16}$$

$$-\frac{d^2\varphi}{d\xi^2}=4\pi\sum_\alpha q_\alpha\int dw f^e_\alpha\ , \tag{17}$$

and

$$w\frac{\partial f^o_\alpha}{\partial\xi}-\frac{q_\alpha}{m_\alpha}\frac{d\varphi}{d\xi}\frac{\partial f^o_\alpha}{\partial w}=0\ , \tag{18}$$

$$0=4\pi\sum_\alpha q_\alpha\int dw\, wf^o_\alpha\ . \tag{19}$$

Any space-dependent equilibrium solution of Eqs. (16)–(19) that is nearby the velocity-shifted equilibrium $F^v_\alpha(w)=F_\alpha(w+v)$ will be a traveling wave nearby the equilibrium $F_\alpha(u)$ in the original frame. Thus the transformation to the wave frame allows us to consider a time-independent problem. We initially concentrate on Eqs. (16) and (17) because these determine the most important quantitative properties of small-amplitude nonlinear waves.

Recall that the Vlasov equation (16) states that the total time derivative of f^e_α vanishes when evaluated along any actual particle trajectory. Thus it is possible to satisfy Eq. (16) by writing the distribution functions f^e_α as smooth functions g^e_α of the conserved single-particle energy $\mathcal{E}_\alpha=\frac{1}{2}m_\alpha w^2+q_\alpha\varphi$, or $f^e_\alpha(\xi,w)=g^e_\alpha(\mathcal{E}_\alpha)$ [12]. Since we are interested in exact *small-amplitude* solutions to Eqs. (16) and (17), however, we must be careful to choose g^e_α so that, when φ is small, $f^e_\alpha(\xi,w)$ is but a small deviation from the even part $F^{v,e}_\alpha$ of the shifted equilibrium [18]. The appropriate definition is

$$g^e_\alpha(\mathcal{E}_\alpha)\equiv g^{v,e}_\alpha(\mathcal{E}_\alpha)=F^{v,e}_\alpha[(2\mathcal{E}_\alpha/m_\alpha)^{1/2}]\ , \tag{20}$$

since this reduces to $F_\alpha^{v,e}$ for $\varphi(\xi)=0$. [Equation (20) defines g_α^e only for $\mathcal{E}_\alpha \ge 0$. Since $\mathcal{E}_\alpha$ can be negative when $\varphi(\xi)\neq 0$, we must extend the definition of g_α^e to include negative values of its argument. Any smooth non-negative extension will suffice [13].]

By substituting this representation for f_α^e into Eq. (17), we obtain a nonlinear differential equation for $\varphi(\xi)$;

$$\frac{d^2\varphi}{d\xi^2} = -\frac{d}{d\varphi} A_v(\varphi) , \tag{21}$$

where the "mechanical potential" $A_v(\varphi)$ is

$$A_v(\varphi)=4\pi \sum_\alpha q_\alpha \int_0^\varphi d\varphi' \int_{-\infty}^{\infty} dw\, g_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2+q_\alpha\varphi') . \tag{22}$$

Equation (21) has the form of Newton's equation, and thus possesses a first integral $H=\frac{1}{2}(d\varphi/d\xi)^2+A_v(\varphi)$. In fact, by analogy with the classical mechanics of a single particle, with ξ corresponding to the time variable, the set of solutions of Eqs. (16) and (17) is determined entirely by the shape of $A_v(\varphi)$, which, in turn, depends both upon the phase velocity v and the distribution functions $F_\alpha(u)$ of the underlying plasma equilibrium. Before analyzing Eq. (21), we first return briefly to Eqs. (18) and (19) for the odd parts of the distribution functions. In BGK form, we may express $f_\alpha^0(\xi,w)$ as

$$f_\alpha^o(\xi,w)=\begin{cases} g_\alpha^o(\mathcal{E}_\alpha), & w\ge 0 \\ -g_\alpha^o(\mathcal{E}_\alpha), & w<0 , \end{cases} \tag{23}$$

where again, since we are interested in near-equilibrium solutions, we choose g_α^o so that $f_\alpha^o(\xi,w)$ reduces, when $\varphi(\xi)=0$, to the odd part $F_\alpha^{v,o}(w)$ of the velocity-shifted equilibrium. The choice for g_α^o analogous to that for g_α^e,

$$g_\alpha^o(\mathcal{E}_\alpha)\equiv g_\alpha^{v,o}(\mathcal{E}_\alpha)=F_\alpha^{v,o}((2\mathcal{E}_\alpha/m_\alpha)^{1/2}) \tag{24}$$

is not quite right, however, since particle trapping in the electrostatic potential of the wave requires $f_\alpha(\xi,w)$ to be even in w in the neighborhood of $w=0$ ($u=v$). This follows from the condition that $f_\alpha(\xi,w)$ be continuous along the line $w=0$, for there $\mathcal{E}_\alpha=q_\alpha\varphi$, and Eq. (23) implies $f_\alpha^o(\xi,0)=g_\alpha^o(q_\alpha\varphi)=-g_\alpha^o(q_\alpha\varphi)=0$. Since this condition must hold over the entire range of values of φ, we must smoothly modify the definition of Eq. (24) so that $g_\alpha^o(\mathcal{E}_\alpha)=0$ for $\mathcal{E}_\alpha \le |q_\alpha\varphi|_{\max}$. Equation (18) is then satisfied, and Eq. (19) becomes a constraint:

$$\sum_\alpha q_\alpha \int_0^\infty dw\, w g_\alpha^o(\mathcal{E}_\alpha)=0 . \tag{25}$$

The details of the modification procedure for g_α^o can be found in Ref. [13]. It is shown there that, for any solution $\varphi(\xi)$ of Eq. (21), the freedom involved in the modification of g_α^o can always be exploited so as to ensure that (i) the constraint of Eq. (25) is satisfied, (ii) the distribution functions f_α are non-negative, and (iii) the odd parts f_α^o uniformly approach those of the equilibrium $F_\alpha^{v,o}(w)$ as $\varphi(\xi)\to 0$. Since the details involved in the demonstration of properties (i)–(iii) are rather technical and, from a physical perspective, not terribly illuminating, we shall omit them here. The important point is that corresponding to any solution of Eqs. (16) and (17) is at least one, and usually many, physically reasonable solutions of Eqs. (18) and (19). In what follows we shall therefore ignore Eqs. (18) and (19) and focus entirely on Eqs. (16) and (17), which determine the form of $\varphi(\xi)$ for any near-equilibrium uniformly translating wave.

Returning to Eq. (21) and now explicitly considering BGK waves of small amplitude, we expand $A_v(\varphi)$ to obtain

$$A_v(\varphi)=\tfrac{1}{2}A_v^{(2)}\varphi^2+\frac{1}{3!}A_v^{(3)}\varphi^3+\frac{1}{4!}A_v^{(4)}\varphi^4+o(\varphi^4) , \tag{26}$$

where, from Eq. (22) and the definition of $g_\alpha^{v,e}$,

$$A_v^{(i)}=\frac{d^i}{d\varphi^i}A_v(\varphi)|_{\varphi=0} = 4\pi\sum_\alpha \frac{q_\alpha^i}{m_\alpha^{i-1}}\int_{-\infty}^{\infty} dw \left[\frac{1}{w}\frac{d}{dw}\right]^{i-1} F_\alpha^{v,e}(w) . \tag{27}$$

The first two terms in the Taylor series of $A_v(\varphi)$ vanish: $A_v^{(0)}=0$ from Eq. (22), while

$$A_v^{(1)}=4\pi\sum_\alpha q_\alpha \int dw\, F_\alpha^{v,e}(w)=0 , \tag{28}$$

since this is the charge density of the Vlasov equilibrium $F_\alpha^v(w)$. Hence the shape of $A_v(\varphi)$ near $\varphi=0$, and thus the form and velocity of the small-amplitude traveling wave solutions, is typically determined by the coefficient $A_v^{(2)}$ of the quadratic term in Eq. (26).

If $A_v^{(2)}$ is positive, there exists a well of finite depth with minimum at $\varphi=0$, and, by analogy with particle motion in a confining well, a corresponding set of periodic solutions $\varphi(\xi)$ which represent periodic traveling waves $\varphi(x-vt)$ in the laboratory frame. Thus at any velocity v where $A_v^{(2)}>0$ there exist spatially periodic traveling waves of arbitrarily small but constant amplitude [13–16]. In the small-amplitude limit, writing $A_v^{(2)}$ as $\kappa^2(v)$ to conform with earlier notation, Eq. (21) becomes

$$\frac{d^2\varphi}{d\xi^2}+\kappa^2(v)\varphi=0 , \tag{29}$$

so that the wave solutions are sinusoidal with wave number $k=\kappa(v)$ $(=\sqrt{A_v^{(2)}})$ and frequency $\omega=\kappa(v)v$. [If the coefficient $\kappa^2(v)=A_v^{(2)}$ vanishes or is negative, then Eq. (21) possesses other types of small-amplitude wave solutions, such as solitary waves [14–16]. We shall not make further reference to such nonperiodic solutions in this paper.]

The relation $k=\kappa(v)$ is of particular importance since when squared it may be written

$$k^2=\kappa^2(v) = 4\pi\sum_\alpha q_\alpha \int dw \frac{1}{w}\frac{d}{dw}F_\alpha^{v,e} = 4\pi\sum_\alpha q_\alpha \mathrm{P}\int du \frac{F_\alpha'}{u-v} , \tag{30}$$

which, for $v=\omega/k$, is just the Vlasov dispersion relation

Eq. (11). In the linear theory of weakly damped waves the Vlasov dispersion relation gives the relationship between frequency and wave number for weakly damped waves. Thus, to every weakly damped wave solution of the linear theory there corresponds an exact nonlinear and undamped small-amplitude BGK wave with the same frequency and wave number. These waves do not appear in the traditional linearized analyses [1,9,10] of Eqs. (6)–(8), since those analyses approximate the derivative $\partial f_\alpha/\partial u$ in the Vlasov equation by dF_α/du, and therefore require the condition

$$\left|\frac{\partial h_\alpha}{\partial u}\right| \ll \left|\frac{dF_\alpha}{du}\right| , \tag{31}$$

where $h_\alpha = f_\alpha - F_\alpha$. For the BGK waves discussed here, however, particle trapping in the electrostatic potential of the wave demands that $\partial f_\alpha/\partial u|_{u=v}=0$ or, equivalently, that

$$\left.\frac{\partial h_\alpha}{\partial u}\right|_{u=v} = -\left.\frac{dF_\alpha}{du}\right|_{u=v} , \tag{32}$$

in violation of the above condition. Equation (32) follows from the requirement of continuity of f_α along the line $u=v$ [see the discussion following Eq. (24)], and expresses the fact that f_α remains a locally even function of u about $u=v$, even as h_α goes to zero. Thus these waves do not behave as predicted by the linear theory since they violate the conditions for its applicability [13]. Physically, the existence of this large class of small-amplitude BGK waves implies that small-amplitude plasma waves that arise as a result of weak perturbing influences can persist, even in so-called linearly stable plasmas.

III. FAILURE OF LINEAR SUPERPOSITION

The nonlinearity involved in the phenomenon of particle trapping is somewhat peculiar, since its effects do not become negligible even as the wave amplitude approaches zero. This behavior also spoils any attempt to build multiple-wave solutions from single waves by linear superposition. To see this, suppose that $(f_\alpha^{(1)}, \varphi^{(1)})$, and $(f_\alpha^{(2)}, \varphi^{(2)})$ are the distribution functions and electric potentials corresponding to BGK waves with velocities v_1 and v_2, where $h_\alpha^{(i)} = f_\alpha^{(i)} - F_\alpha$ are small deviations of the distribution functions from those of a Vlasov equilibrium F_α. The linearly superimposed state then is

$$\begin{aligned} f_\alpha^L &= F_\alpha + h_\alpha^{(1)} + h_\alpha^{(2)} , \\ \varphi^L &= \varphi^{(1)} + \varphi^{(2)} , \end{aligned} \tag{33}$$

in which the distribution functions are given by their equilibrium values plus the deviations $h_\alpha^{(1)}$ and $h_\alpha^{(2)}$ corresponding to each separate wave. [The expression $f_\alpha^L = f_\alpha^{(1)} + f_\alpha^{(2)}$ would clearly be incorrect since it would include the background equilibrium distribution functions $F_\alpha(u)$ twice.] The linear superposition given by Eq. (33) satisfies both the Poisson and Ampère equations [Eqs. (7) and (8)] exactly by virtue of their linearity, while the Vlasov equation becomes

$$\left[\frac{d}{dt}\right]_L f_\alpha^L = -\frac{q_\alpha}{m_\alpha}\left[\frac{\partial\varphi^{(1)}}{\partial x}\frac{\partial h_\alpha^{(2)}}{\partial u} + \frac{\partial\varphi^{(2)}}{\partial x}\frac{\partial h_\alpha^{(1)}}{\partial u}\right] , \tag{34}$$

where $(d/dt)_L$ denotes the time derivative evaluated along particle trajectories in the superimposed field $\varphi^L(x,t)$. For small-amplitude waves, the deviations $h_\alpha^{(i)}$ and potentials $\varphi^{(i)}$ each are of first order in a small parameter ϵ, as is the left side of Eq. (34). That the right side of Eq. (34) appears to be second order in ϵ suggests that (f_α^L, φ^L) is an approximate superimposed solution that grows more exact as $\epsilon \to 0$. However, since BGK waves trap particles even at small amplitudes, as discussed in Sec. II, the single-wave distribution functions $f_\alpha^{(i)}$ must each satisfy for all ϵ the condition given by Eq. (32) in accordance with the formation of a small plateau at the wave velocity in the space-averaged distribution functions. However, this implies that in the neighborhood of either phase velocity v_1 or v_2, the right side of Eq. (34) actually is only first order in ϵ, and thus remains important even as the wave amplitudes become very small. Thus the linear superposition fails due to the unusual nonlinearity involved in the phenomenon of particle trapping.

Thus we see that the simple linear superposition fails to give a new solution even for waves of arbitrarily small amplitude. But the manner in which this superposition fails is instructive; for the trouble comes not from the linear Poisson or Ampère equations, but from the Vlasov equation, which suggests that it may be possible to find a solution of the slightly more general form:

$$f_\alpha^{(+)} = h_\alpha^{(1)} + h_\alpha^{(2)} + f_\alpha^{\text{int}} , \tag{35}$$

$$\varphi^{(+)} = \varphi^{(1)} + \varphi^{(2)} , \tag{36}$$

where f_α^{int} is an interaction term. Such a term would at once modify the distribution functions in the neighborhood of each component wave velocity, so that $f_\alpha^{(+)}$ would satisfy the Vlasov equation uniformly through first order in ϵ, and yet would contribute to the charge and current densities only at higher than first order so that the Poisson and Ampère equations would remain satisfied in the small-amplitude limit. If a solution of this type could be found, it would then provide a nonlinear superposition principle for small-amplitude BGK waves in which the electric fields superimpose linearly while the distribution functions combine according to the nonlinear prescription given in Eq. (35).

If we substitute Eqs. (35) and (36) into the Vlasov, Poisson, and Ampère equations, we find three conditions on f^{int} which must be satisfied if $(f_\alpha^{(+)}, \varphi^{(+)})$ is to be a uniformly valid solution through first order in ϵ. These conditions are

$$\frac{\partial f_\alpha^{\text{int}}}{\partial t} + u\frac{\partial f_\alpha^{\text{int}}}{\partial x} - \frac{q_\alpha}{m_\alpha}\frac{\partial\varphi^{(+)}}{\partial x}\frac{\partial f_\alpha^{\text{int}}}{\partial u} = \frac{q_\alpha}{m_\alpha}\left[\frac{\partial\varphi^{(1)}}{\partial x}\frac{\partial h_\alpha^{(2)}}{\partial u} + \frac{\partial\varphi^{(2)}}{\partial x}\frac{\partial h_\alpha^{(1)}}{\partial u}\right] + o(\epsilon) , \tag{37}$$

$$\sum_\alpha q_\alpha \int du\, f_\alpha^{\text{int}} = o(\epsilon) \;, \tag{38}$$

$$\sum_\alpha q_\alpha \int du\, u f_\alpha^{\text{int}} = o(\epsilon) \;. \tag{39}$$

The first of these is an inhomogeneous Vlasov equation for f_α^{int}, which in this case is linear since we neglect terms of higher than first order in the electric potential. The second and third equations express constraints on the interaction term f_α^{int} necessary to avoid altering the first-order field, thus assuring self-consistency through first order.

To solve the inhomogeneous Vlasov equation for f_α^{int} requires integrating along the characteristic curves, which in this case are the particle trajectories in the linearly superimposed field $\varphi^{(+)}=\varphi^{(1)}+\varphi^{(2)}$. Thus at least an approximate knowledge of these trajectories is required to find f_α^{int}. Supposing these trajectories are known, then Eq. (37) can be inverted to give f_α^{int} apart from its arbitrary initial solution $f_\alpha^{\text{int}}(x,u,t=0)$. Upon substituting f_α^{int} into the constraints [Eqs. (38) and (39)] the strategy is then to choose the initial distribution $f_\alpha^{\text{int}}(x,u,t=0)$ in order to satisfy these constraints. If this can be accomplished, then an approximate self-consistent nonlinear solution describing superimposed BGK waves is thereby obtained.

In Section V, we shall construct a uniformly approximate two-wave solution, although we shall not use the exact procedure suggested above, opting instead for a related but more direct method that does not require explicit calculation of f_α^{int}. Nevertheless, the preceding discussion should make it clear that the critical ingredient in constructing a superimposed two-wave solution, if such a solution exists, is understanding the nature of particle trajectories in the superposed field $\varphi^{(+)}$.

IV. PARTICLE DYNAMICS

In the case of a uniformly translating wave $\varphi=\varphi(x-vt)$, it is the existence of the single-particle energy invariants $\mathcal{E}_\alpha=m_\alpha(u-v)^2/2+q_\alpha\varphi(x-vt)$ which gives the BGK method its power, since it allows the Vlasov equation to be solved exactly—one merely writes $f_\alpha(x,u,t)=g_\alpha(\mathcal{E}_\alpha)$ for any set of smooth, non-negative functions g_α. In a more general field $\varphi(x,t)$ the situation is far more complicated. If the field $\varphi(x,t)$ has a nontrivial time dependence that cannot be transformed away by a simple shift in reference frame, then the single-particle Hamiltonian $H_\alpha=\mathcal{E}_\alpha=m_\alpha u^2/2+q_\alpha\varphi(x,t)$ is not conserved since $\partial H_\alpha/\partial t\neq 0$. Accordingly, $f_\alpha(x,u,t)=g_\alpha(\mathcal{E}_\alpha)$ does not satisfy the Vlasov equation since the energies $\mathcal{E}_\alpha$ are no longer constant along particle trajectories. The failure of the simple linear superposition (f_α^L,φ^L) given by Eq. (33) is therefore not surprising, since the distribution functions are written there in terms of quantities that are not invariant in the field φ^L. A more appropriate procedure for developing a two-wave solution is first to find invariant quantities for the two-wave field, and then use these invariants to construct distribution functions. Thus we search for the proper generalizations of the two single-wave energy invariants $\mathcal{E}_\alpha^{(i)}=m_\alpha(u-v_i)^2/2+q_\alpha\varphi^{(i)}(x,t)$ to the two-wave case.

Consider the Hamiltonian system with Hamiltonian

$$H_\alpha(x,p,t)=p^2/2m_\alpha+\epsilon\varphi^{(1)}(x,t)+\epsilon\varphi^{(2)}(x,t) \;, \tag{40}$$

which describes the dynamics of a charged particle in the field of two small-amplitude electrostatic waves, $\varphi^{(1)}(x,t)=-\bar\varphi_1\cos(k_1x-\omega_1t)$ and $\varphi^{(2)}(x,t)=-\bar\varphi_2\cos(k_2x-\omega_2t)$, where $\bar\varphi_1$, $\bar\varphi_2$ are $O(1)$. For definiteness we shall consider the case in which $(v_1=\omega_1/k_1)>(v_2=\omega_2/k_2)$. For $\epsilon=0$, the Hamiltonian $H_\alpha(x,p,t)$ reduces to the free particle Hamiltonian for which the dynamics is of course fully integrable. For $\epsilon\neq 0$, the Hamiltonian system corresponding to Eq. (40) is only nearly integrable, and, in addition to invariant curves, the x-p phase plane is riddled with stochastic regions associated with the transverse intersections of the invariant manifolds of hyperbolic fixed points [19]. It is therefore impossible to find global invariants for this system, i.e., to find smooth invariant functions that are well behaved over the entire phase plane when $\epsilon\neq 0$. Nevertheless, approximate invariants, valid over restricted regions of the plane, can be developed via perturbation methods. This approach will suffice here, however, since the stochastic regions occupy an area of the x-p plane that is exponentially small for small wave amplitudes, and the effects of these regions can thus be ignored in this limit.

If Hamiltonian perturbation theory is applied straightforwardly in order to find the modified trajectories when $\epsilon\neq 0$, one immediately finds that at first order the expressions for $x(t)$ and $p(t)$ contain resonant factors $(p_0-m_\alpha v_1)^{-1}$ and $(p_0-m_\alpha v_2)^{-1}$ which become singular as the unperturbed velocity $u_0=p_0/m_\alpha$ approaches either wave velocity v_1 or v_2. These are the so-called resonant denominators associated with the resonant interaction of the particle with the primary waves. For the moment consider velocities in the neighborhood of v_1. In order to avoid the appearance of the singular factor $(p_0-m_\alpha v_1)^{-1}$, and thus obtain an expression useful for velocities near v_1 [although $(p_0-m_\alpha v_2)^{-1}$ will still appear], it is necessary first to perform a preparatory canonical transformation of the system before applying perturbation theory [19]. The transformation is little more than a Galilean shift into a frame of reference moving with the first wave—nevertheless, it is useful to employ a formal approach so as to ensure that the transformation is canonical.

We define a function $F_2(x,J,t)$ which generates a canonical transformation from the old variables (x,p) to new canonical variables (θ,J) as

$$F_2(x,J,t)=J(k_1x-\omega_1t)+m_\alpha v_1 x-\tfrac{1}{2}m_\alpha v_1^2 t \;. \tag{41}$$

In Goldstein's notation [20] "F_2" is a generating function that depends upon old coordinates and new momenta. The transformation is canonical when defined through the relations

$$\theta=\frac{\partial F_2}{\partial J}=kx-\omega t\ ,$$
$$p=\frac{\partial F_2}{\partial x}=kJ+\beta\ , \qquad (42)$$

on the condition that the new Hamiltonian is related to the old as

$$\tilde{H}(\theta,J,t)=H(x(\theta,J),p(\theta,J),t)+\frac{\partial F_2}{\partial t}$$
$$=\frac{(k_1J+m_\alpha v_1)^2}{2m_\alpha}-\omega J-\tfrac{1}{2}m_\alpha v_1^2-\epsilon q_\alpha\bar{\varphi}_1\cos\theta$$
$$-\epsilon q_\alpha\bar{\varphi}_2\cos\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]\ . \qquad (43)$$

This expression may be simplified algebraically (dropping the *tilde* notation) to

$$H(\theta,J,t)=H_0(\theta,J,t)+\epsilon H_1(\theta,J,t)\ , \qquad (44)$$

where

$$H_0=\frac{k_1^2}{2m_\alpha}J^2\ ,$$
$$H_1=-q_\alpha\bar{\varphi}_1\cos\theta-q_\alpha\bar{\varphi}_2\cos\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]\ , \qquad (45)$$

which completes the preparatory transformation.

We shall use the Lie Transform formalism [19] to apply perturbation theory to the system described by the Hamiltonian in Eq. (44). In this formalism, the relation

$$\frac{d\bar{x}(x,\epsilon)}{d\epsilon}=[\bar{x},w] \qquad (46)$$

induces a family of canonical transformations, parametrized by ϵ, from the old variables $x=(\theta,J)$ to new variables $\bar{x}(x,\epsilon)=(\bar{\theta}(\theta,J,\epsilon),\bar{J}(\theta,J,\epsilon))$. Here $[f,g]$ is the Poisson bracket operation

$$[f,g]=\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial J}-\frac{\partial f}{\partial J}\frac{\partial g}{\partial\theta}\ , \qquad (47)$$

and the Lie generating function $w=w(x,\epsilon)$ depends on ϵ as well as on the old variables. The identity transformation is given by the "initial condition" $\bar{x}(x,\epsilon=0)=x$. By expanding both $\bar{x}(x,\epsilon)$ and $w(x,\epsilon)$ as power series in ϵ, i.e., $\bar{x}=\bar{x}_0+\epsilon\bar{x}_1+O(\epsilon^2)$ and $w=w_1(x)+O(\epsilon)$, it is straightforward to obtain the first-order relation between the new and old coordinates as

$$\bar{x}(x,\epsilon)=x+\epsilon[x,w_1(x)]+O(\epsilon^2)\ . \qquad (48)$$

To first order, the perturbation calculation proceeds by finding $w_1(\theta,J)$, which satisfies the equation

$$\frac{\partial w_1}{\partial t}+[w_1,H_0]=\bar{H}_1-H_1\ . \qquad (49)$$

We are free to choose $\bar{H}_1$, the first term in an expansion of the new Hamiltonian, for convenience. To motivate our choice, observe that the Poisson bracket in Eq. (49) reduces to

$$[w_1,H_0]=\frac{k_1^2}{m_\alpha}J\frac{\partial w_1}{\partial\theta}\ , \qquad (50)$$

which vanishes on the line $J=0$. Thus along this line the equation for w_1 is

$$\frac{\partial w_1}{\partial t}=\bar{H}_1+q_\alpha\bar{\varphi}_1\cos\theta$$
$$+q_\alpha\bar{\varphi}_2\cos\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]\ . \qquad (51)$$

Since the term containing $\cos\theta$ does not depend explicitly on t, it follows that for $J=0$ the function w_1 must contain a term proportional to $t\cos\theta$. The appearance of this secular term, which becomes unbounded with time, would limit the usefulness of any result to short times. Thus, to avoid this secularity, we choose $\bar{H}_1$ to eliminate the term in H_1 that is its source. That is, we choose

$$\bar{H}_1(\theta,J)=-q_\alpha\bar{\varphi}_1\cos\theta\ , \qquad (52)$$

in which case the new Hamiltonian is

$$\bar{H}(\bar{\theta},\bar{J})=\bar{H}_0(\bar{\theta},\bar{J})+\epsilon\bar{H}_1(\bar{\theta},\bar{J})+O(\epsilon^2)$$
$$=\frac{k_1^2}{2m_\alpha}\bar{J}^2-\epsilon q_\alpha\bar{\varphi}_1\cos\bar{\theta}+O(\epsilon^2)\ . \qquad (53)$$

The equation for w_1 [Eq. (49)] then becomes

$$\frac{\partial w_1}{\partial t}+\frac{k_1^2}{m_\alpha}J\frac{\partial w_1}{\partial\theta}=q_\alpha\bar{\varphi}_1\cos\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]\ , \qquad (54)$$

for which a solution is

$$w_1=\frac{q_\alpha\bar{\varphi}_1}{\dfrac{k_1k_2}{m_\alpha}J+k_2(v_1-v_2)}\sin\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]\ . \qquad (55)$$

The Lie generating function w_1 contains all information concerning the first-order solution, since it generates the first-order canonical transformation through Eq. (48). Written out explicitly, we have

$$\bar{\theta}=\theta-\epsilon[w_1,\theta]+O(\epsilon^2)$$
$$=\theta-\epsilon\frac{k_1k_2}{m_\alpha}\frac{q_\alpha\bar{\varphi}_1}{\left[\dfrac{k_1k_2}{m_\alpha}J+k_2(v_1-v_2)\right]^2}$$
$$\times\sin\left[\frac{k_2}{k_1}\theta+k_2(v_1-v_2)t\right]+O(\epsilon^2) \qquad (56)$$

and

$$\bar{J} = J - \epsilon[w_1, J] + O(\epsilon^2)$$

$$= J - \epsilon \frac{k_1 k_2}{m_\alpha} \frac{q_\alpha \bar{\varphi}_1}{\dfrac{k_1^2}{m_\alpha} J + k_1(v_1 - v_2)} \cos\left[\frac{k_2}{k_1}\theta + k_2(v_1 - v_2)t\right] + O(\epsilon^2) . \tag{57}$$

The portion $\bar{H}_0(\bar{\theta},\bar{J}) + \epsilon \bar{H}_1(\bar{\theta},\bar{J})$ of the new Hamiltonian $\bar{H}(\bar{\theta},\bar{J})$ is an invariant through first order in ϵ. For future use it is convenient to write this invariant, which we shall call $\bar{\mathcal{E}}_\alpha^{(1)}$, in terms of the original variables x and u. A little algebra, using Eqs. (42), (53), (56), and (57), gives

$$\bar{H} = \tfrac{1}{2} m_\alpha \left[u - v_1 - \epsilon \frac{q_\alpha \bar{\varphi}_2}{m_\alpha (u - v_2)} \cos(k_2 x - \omega_2 t)\right]^2$$
$$- \epsilon q_\alpha \bar{\varphi}_1 \cos\left[k_1 x - \omega_1 t - \epsilon \frac{k_1}{k_2} \frac{q_\alpha \bar{\varphi}_2}{m_\alpha (u - v_2)^2} \sin(k_2 x - \omega_2 t)\right] + O(\epsilon^2) , \tag{58}$$

which when expanded for small ϵ becomes

$$\bar{H} = \tfrac{1}{2} m_\alpha (u - v_1)^2 - \epsilon q_\alpha \bar{\varphi}_1 \cos(k_1 x - \omega_1 t)$$
$$- \epsilon q_\alpha \frac{u - v_1}{u - v_2} \bar{\varphi}_2 \cos(k_2 x - \omega_2 t) + O\left[\frac{\epsilon^2}{(u - v_2)^2}\right] . \tag{59}$$

As long as we restrict u to values far from v_2, then the terms written as $O[\epsilon^2/(u - v_2)^2]$ in this expression are uniformly of order ϵ^2. In fact, as long as $u - v_2$ is $O(\epsilon^p)$, where $p < 1/2$, these terms are $o(\epsilon)$ and thus higher order than the terms retained explicitly in Eq. (59). The first two terms on the right are recognizable as the single-wave energy invariant $\mathcal{E}_\alpha^{(1)}$ for a particle in the field of the first wave, while the third term gives the lowest-order correction to $\mathcal{E}_\alpha^{(1)}$ due to the influence of the second wave. Thus for u not too close to v_2 we have a first-order invariant for the two-wave field:

$$\bar{\mathcal{E}}_\alpha^{(1)} = \mathcal{E}_\alpha^{(1)} - \epsilon q_\alpha \frac{u - v_1}{u - v_2} \bar{\varphi}_2 \cos(k_2 x - \omega_2 t) . \tag{60}$$

By repeating this entire procedure, but shifting first to a reference frame translating with the second wave, we obtain a corresponding first-order invariant $\bar{\mathcal{E}}_\alpha^{(2)}$ associated with the second wave. The resulting pair of first-order invariants, which reduce for $\epsilon = 0$ to free-particle kinetic energy invariants $\mathcal{E}_\alpha^{(i)} = m_\alpha (u - v_i)^2/2$, are together concisely expressed as

$$\bar{\mathcal{E}}_\alpha^{(i)} = \mathcal{E}_\alpha^{(i)} + \epsilon q_\alpha \frac{u - v_i}{u - v_j} \varphi^{(j)}(x,t) ,$$
$$i,j = 1,2,\text{cyclic} . \tag{61}$$

We would like to piece these invariants together in such a way as to provide a nonsingular invariant covering of the x-u plane. To do this, we observe that these two invariants are equal through first order in ϵ along the time-dependent curve

$$u_\alpha(x,t) = v_{\text{av}} + (2\epsilon/m_\alpha \delta v)[\varphi^{(1)}(x,t) - \varphi^{(2)}(x,t)] , \tag{62}$$

where $v_{\text{av}} = (v_1 + v_2)/2$ and $\delta v = v_1 - v_2$. Although neither $\bar{\mathcal{E}}_\alpha^{(1)}$ nor $\bar{\mathcal{E}}_\alpha^{(2)}$ is a global first-order invariant by virtue of the respective singularities at the phase velocities v_2 and v_1, $\bar{\mathcal{E}}_\alpha^{(1)}$ is nevertheless well behaved for $u \ge u_\alpha(x,t)$, as is $\bar{\mathcal{E}}_\alpha^{(2)}$ for $u \le u_\alpha(x,t)$. Thus $\bar{\mathcal{E}}_\alpha^{(1)}$ and $\mathcal{E}_\alpha^{(2)}$ taken together piecewise in this manner provide the desired invariant covering of the x-u plane for particle species α. These invariants will be the key elements used in Sec. V in the construction of distribution functions for a two-wave state.

The quantities $\bar{\mathcal{E}}_\alpha^{(1)}$ and $\mathcal{E}_\alpha^{(2)}$ only approximately describe the dynamics of particles in the two-wave field. To exhibit the exact dynamics we have plotted in Fig. 1, for the Hamiltonian system corresponding to the Hamiltonian of Eq. (40), numerically generated successive intersections of various particle trajectories with a Poincaré surface of section defined by stroboscopic sampling at times $t_n = nT$, $n = \cdots -1, 0, 1, \ldots$. This figure corresponds to $q_\alpha \bar{\varphi}_1/m_\alpha = q_\alpha \bar{\varphi}_2/m_\alpha = 1$, $\epsilon = 0.1$ for the specific case $k_1 = 2k$, $\omega_1 = 2\omega$, $k_2 = 3k$, and $\omega_2 = -3\omega$. Here $T = 2\pi/\omega$ is the period of the two-wave field, and the particle velocity u is plotted vertically versus the spatial phase $\psi = kx \bmod 2\pi$. The closed invariant curves centered near the component wave velocities $v_1 = \omega/k$ and $v_2 = -\omega/k$ correspond to particles trapped by either of the two individual waves, while the snakelike invariant curves at larger velocities correspond to untrapped particles. Figure 2 shows some of the level curves of the approximate invariants $\bar{\mathcal{E}}_\alpha^{(1)}$ and $\bar{\mathcal{E}}_\alpha^{(2)}$ evaluated on the same Poincaré surface; clearly, $\bar{\mathcal{E}}_\alpha^{(1)}$ and $\bar{\mathcal{E}}_\alpha^{(2)}$ capture the gross features of the particle dynamics, including the first-order resonance structures, although they do not capture the nonintegrable character of the true dynamics which is reflected in the stochastic layers (dark areas corresponding to single trajectories). The contours of the first-order invariants $\bar{\mathcal{E}}_\alpha^{(1)}$ and $\bar{\mathcal{E}}_\alpha^{(2)}$ also do not exhibit small island structures corresponding to higher-order resonances. This

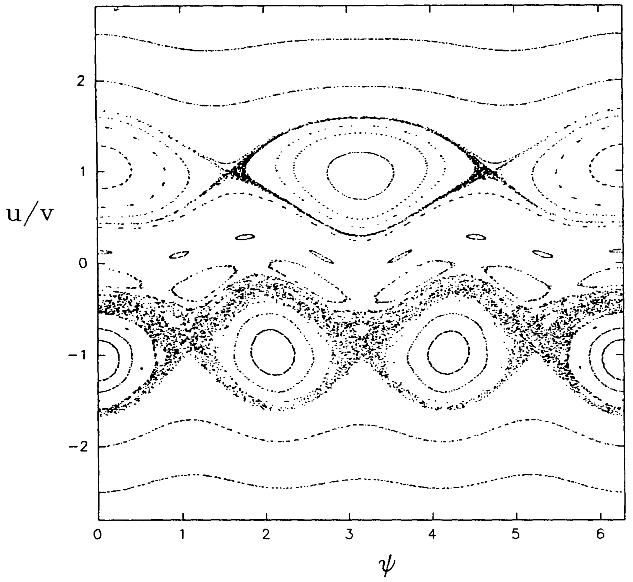

FIG. 1. The orbits of a Poincaré map constructed by stroboscopically sampling numerically generated particle trajectories in the two-wave field $\varphi(x,t)=-\epsilon\cos(2kx-2\omega t)-\epsilon\cos(3kx+3\omega t)$. Phase plane coordinates are (ψ,u), where $\psi=kx \bmod 2\pi$.

higher order structure could be accounted for by continuing the perturbation calculation to higher order, but for our purposes here we shall not require such detail.

V. MULTIPLE-WAVE DISTRIBUTION FUNCTIONS

The approximate invariants $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\overline{\mathcal{E}}_\alpha^{(2)}$ given in Eq. (61) can now be used to construct distribution functions for a two-wave state. If $g_\alpha^{(i)}$ are the BGR functions for single waves such that $f_\alpha^{(i)}(x,u,t)=g_\alpha^{(i)}(\mathcal{E}_\alpha^{(i)})$, then we define the distribution functions $f_\alpha^{(+)}$ for the two-wave state as

$$f_\alpha^{(+)}(x,u,t)=\begin{cases} g_\alpha^{(1)}(\overline{\mathcal{E}}_\alpha^{(1)}) , & u \ge u_\alpha(x,t) \\ g_\alpha^{(2)}(\overline{\mathcal{E}}_\alpha^{(2)}) , & u \le u_\alpha(x,t) . \end{cases} \tag{63}$$

Here $g_\alpha^{(i)}$ is a condensed notation for the pair of functions $\{g_\alpha^{e,(i)},g_\alpha^{o,(i)}\}$ which actually specify a BGK representation as discussed in Sec. II. In other words, to build two-wave distribution functions we use the function $g_\alpha^{(1)}$ of the first wave and the associated invariant $\overline{\mathcal{E}}_\alpha^{(1)}$ above the curve $u_\alpha(x,t)$, and the function $g_\alpha^{(2)}$ of the second wave and $\overline{\mathcal{E}}_\alpha^{(2)}$ below $u_\alpha(x,t)$. We have not arrived at the definition of Eq. (63) by the application of any well-defined methodology. Rather, it is an educated guess which we can shall now show *a posteriori* possesses the requisite properties.

This definition clearly gives distribution functions that, through first order in ϵ, satisfy the Vlasov equation since they are written in terms of $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\overline{\mathcal{E}}_\alpha^{(2)}$. For instance, since the Vlasov operator may be interpreted as the total time derivative evaluated along particle trajectories, we calculate, for $u > u_\alpha(x,t)$ for instance,

$$\begin{aligned}\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}-\frac{q_\alpha}{m_\alpha}\frac{\partial\varphi^{(+)}}{\partial x}\frac{\partial}{\partial u}\right]f_\alpha^{(+)} &= \frac{d}{d\tau}g_\alpha^{\nu_1}(\overline{\mathcal{E}}_\alpha^{(1)}) \\ &= \frac{dg_\alpha^{\nu_1}}{d\eta}(\overline{\mathcal{E}}_\alpha^{(1)})\frac{d}{d\tau}\overline{\mathcal{E}}_\alpha^{(1)} \\ &= o(\epsilon) ,\end{aligned} \tag{64}$$

where $dg_\alpha^{\nu_1}/d\eta$ denotes the derivative of $g_\alpha^{\nu_1}$ with respect to its argument, and the final line holds since $\overline{\mathcal{E}}_\alpha^{(1)}$ is itself a first-order invariant. A similar result holds for $u < u_\alpha(x,t)$ since $\overline{\mathcal{E}}_\alpha^{(2)}$ is invariant. In addition, the Vlasov equation is also satisfied to first-order along $u=u_\alpha(x,t)$ since $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\overline{\mathcal{E}}_\alpha^{(2)}$ meet smoothly there as shown in Fig. 2. This figure, incidentally, also doubles as a plot of the level curves of $f_\alpha^{(+)}$ when evaluated on the Poincaré surface.

As discussed in Sec. III, if these distribution functions are to describe a self-consistent solution to the Vlasov-Poisson-Ampère equations, they must yield charge and current densities $\rho(x,t)$ and $j(x,t)$ which generate the correct self-consistent superimposed electric potential $\varphi^{(+)}=\varphi^{(1)}+\varphi^{(2)}$. Thus the distribution functions of Eq. (63) must satisfy, from Eqs. (7) and (8),

$$-\frac{\partial^2\varphi^{(+)}}{\partial x^2}=4\pi\sum_\alpha q_\alpha\int_{-\infty}^{\infty}du\, f_\alpha^{(+)}+o(\epsilon) \tag{65}$$

and

$$\frac{\partial^2\varphi^{(+)}}{\partial x\,\partial t}=4\pi\sum_\alpha q_\alpha\int_{-\infty}^{\infty}du\, uf_\alpha^{(+)}+o(\epsilon) . \tag{66}$$

That the above conditions for self-consistency are indeed

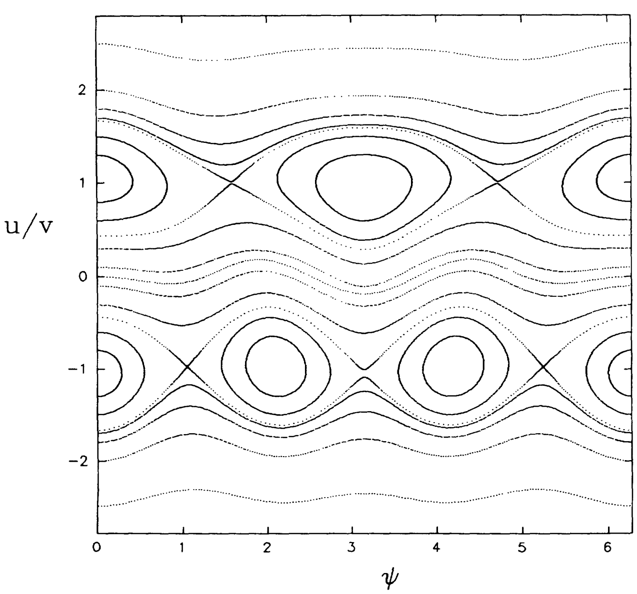

FIG. 2. Level curves of the first-order invariants $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\overline{\mathcal{E}}_\alpha^{(2)}$, or equivalently, of the two-wave distribution function $f_\alpha^{(+)}$, when evaluated on the Poincaré section. Curves were generated for $u \ge u(x,t)$ with $\overline{\mathcal{E}}_\alpha^{(1)}$, and for $u < u(x,t)$ with $\overline{\mathcal{E}}_\alpha^{(2)}$.

satisfied through first order can be demonstrated by a lengthy calculation using the detailed definitions of the single-wave BGK functions $g_\alpha^{(i)}$ for small-amplitude waves which were discussed briefly in Sec. II, and in more detail in Ref. [13]. This calculation has been done and is available [16]. But this result can also be obtained more simply. We saw in Sec. III that the linear theory is adequate outside the regions where particles are trapped by each individual wave. Thus, outside these regions in which u satisfies $m_\alpha(u-v_i)^2/2\le 2\epsilon|q_\alpha\overline{\varphi}_i|$ for $i=1,2$, the functions $f_\alpha^{(+)}$ agree through first-order with the linearly superimposed distribution functions $f_\alpha^L=F_\alpha(u)+h_\alpha^{(1)}(\mathcal{E}_\alpha^{(1)})+h_\alpha^{(2)}(\mathcal{E}_\alpha^{(2)})$. On the other hand, since both $f_\alpha^{(+)}$ and f_α^L differ from the equilibrium F_α by terms of order ϵ, the difference between $f_\alpha^{(+)}$ and f_α^L inside the trapping regions satisfies $f_\alpha^{(+)}-f_\alpha^L=O(\epsilon)$. And since the widths in velocity $\Delta u^{(i)}$ of the trapping regions satisfy $\Delta u^{(i)}=O(\epsilon^{1/2})$, the integrals in Eqs. (65) and (66) can be calculated correctly to first order in ϵ simply by replacing $f_\alpha^{(+)}$ everywhere with f_α^L. The errors accrued in the integrations over the trapping regions as a result of this replacement will be only $O(\epsilon^{3/2})$. Since each component wave independently satisfies the Poisson and Ampère equations, which requires that each pair (ω_1,k_1) and (ω_2,k_2) be a root of the Vlasov dispersion relation of Eq. (11), integration over f_α^L thus verifies that $f_\alpha^{(+)}$ is a self-consistent solution through first order in ϵ.

VI. NONLINEAR SUPERPOSITION

Thus, we have obtained a self-consistent superimposed solution which describes a plasma state containing two small-amplitude periodic BGK waves. This solution embodies a nonlinear superposition principle, in which the single-wave potentials are superimposed linearly, $\varphi^L=\varphi^{(1)}+\varphi^{(2)}$, while the distribution functions $f_\alpha^{(+)}$ are constructed from the single-wave distributions by the nonlinear rule given in Eq. (63). The conditions under which this superposition principle holds may be surmised by more detailed consideration of $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\mathcal{E}_\alpha^{(2)}$. Over the regions in which they are used in the definition of Eq. (63), these quantities are approximate invariants, with errors that are at most $O(\epsilon^2/\delta v^2)$. For $\overline{\mathcal{E}}_\alpha^{(1)}$ and $\overline{\mathcal{E}}_\alpha^{(2)}$ to be true first-order invariants thus requires that $\delta v=O(\epsilon^s)$, where $s<\frac{1}{2}$. Reverting to the physical wave amplitudes $\varphi^{(1)}$ and $\varphi^{(2)}$ (each of which is of order ϵ), this condition takes the form

$$\varphi^{(1)},\varphi^{(2)}\ll(m_\alpha/q_\alpha)\delta v^2\ , \tag{67}$$

which must hold for all species in the plasma. Physically this reflects the fact that two waves interact less strongly the larger their relative phase velocity, since for large δv particles trapped in one wave feel only a high frequency perturbation from the passing field of the other, and thus their motion is on average affected very little.

Extension of the preceding development to the nonlinear superposition of N small-amplitude waves is straightforward and leads to a simple generalization of Eq. (63). Suppose that the velocities of the N waves are arranged in ascending order $v_1<v_2\ldots<v_{N-1}<v_N$, and that the electric potential associated with each wave is $\varphi^{(i)}=-\epsilon\overline{\varphi}_i\cos(k_ix-\omega_it)$. By direct extension of the developments for the two-wave case, the N first-order invariant quantities $\overline{\mathcal{E}}_\alpha^{(i)}$, $i=1,2,\ldots,N$ for particle motion in the N-wave electrostatic field $\varphi^{(N)}(x,t)=\sum_i^N\varphi^{(i)}(x,t)$ are

$$\overline{\mathcal{E}}_\alpha^{(i)}=\tfrac{1}{2}m_\alpha(u-v_i)^2-\epsilon q_\alpha\overline{\varphi}_i\cos(k_ix-\omega_it)$$

$$-\epsilon q_\alpha\sum_{j\neq i}\frac{u-v_i}{u-v_j}\overline{\varphi}_j\cos(k_jx-\omega_jt)\ . \tag{68}$$

As in the two-wave case, it is possible to use these invariants to construct a set of distribution functions $f_\alpha^{(N)}$ for the superimposed state that satisfy the Vlasov equation uniformly through first order in ϵ. To do this we first define a set of $N-1$ boundaries $u_{i,i+1}(x,t)$, $i=1,2,N-1$, which will serve to delimit regions over which particular members of the set of N invariants $\overline{\mathcal{E}}_\alpha^{(i)}$, $i=1,2,\ldots,N$ will be used to define the functions $f_\alpha^{(N)}$. Each $u_{i,i+1}(x,t)$ is defined as a curve between the velocities v_i and v_{i+1} along which $\overline{\mathcal{E}}_\alpha^{(i)}$ and $\overline{\mathcal{E}}_\alpha^{(i+1)}$ agree to first order in ϵ. (For simplicity we have omitted the index α which should appear on each $u_{i,i+1}$ since these curves generally are different for each particle species.) A straightforward calculation using Eq. (68) gives

$$u_{i,i+1}(x,t)=\tfrac{1}{2}(v_i+v_{i+1})+2\epsilon\frac{q_\alpha}{m_\alpha}\sum_{j=1}^{N}\frac{\overline{\varphi}_j\cos(k_jx-\omega_jt)}{v_i+v_{i+1}-2v_j}\ . \tag{69}$$

Thus, we define the N-wave distribution functions as

$$f_\alpha^{(N)}(x,u,t)=\begin{cases} g_\alpha^{v_1}(\overline{\mathcal{E}}_\alpha^{(1)})\ , & u\le u_{1,2}(x,t)\\ g_\alpha^{v_i}(\overline{\mathcal{E}}_\alpha^{(i)})\ , & u_{i-1,i}(x,t)\le u\le u_{i,i+1}(x,t)\ ,\quad i=2,3,\ldots,N-1\\ g_\alpha^{v_N}(\overline{\mathcal{E}}_\alpha^{(N)})\ , & u\ge u_{N-1,N}(x,t)\ .\end{cases} \tag{70}$$

The demonstration of the self-consistency of the state described by these distribution functions follows the same line of argument used in the two-wave case. There is nothing particularly problematic in the transition from two to N waves, at least if N does not become too large. To estimate roughly how large N can become before difficulties arise, we must consider two separate conditions. First, the various wave velocities and amplitudes must be such that the quantities $\overline{\mathcal{E}}_\alpha^{(i)}$ given in Eq. (68) remain true first-order invariants in the N-wave field. Second, the sum of the $O(\epsilon^{3/2})$ residual contributions to the electric potential which come from the N resonant regions centered at the wave velocities must be $o(\epsilon)$. We consider these two conditions in order below.

As in the two-wave case, the invariant $\overline{\mathcal{E}}_\alpha^{(i)}$ is identified as a part of the new Hamiltonian that is obtained by carrying out the perturbation calculation through first order in ϵ after shifting to a frame of reference translating at velocity v_i. The full form of this new Hamiltonian in the N-wave case is

$$\overline{H}=\tfrac{1}{2}m_\alpha(u-v_i)^2-\epsilon q_\alpha\overline{\varphi}_i\cos(k_ix-\omega_it)$$
$$-\epsilon q_\alpha\sum_{j\neq i}\frac{u-v_i}{u-v_j}\overline{\varphi}_j\cos(k_jx-\omega_jt)$$
$$+\sum_{j\neq i}O\left[\frac{\epsilon^2}{(u-v_j)^2}\right]$$
$$=\overline{\mathcal{E}}_\alpha^{(i)}+\sum_{j\neq i}O\left[\frac{\epsilon^2}{(u-v_j)^2}\right], \tag{71}$$

which has as a special case the result given by Eq. (59) for the two-wave case. If the quantity $\overline{\mathcal{E}}_\alpha^{(i)}$ is to be invariant through first order over the region in which it is used in the definition of $f_\alpha^{(N)}$, given by Eq. (70), then it is necessary that the remaining term in Eq. (71) be $o(\epsilon)$ when evaluated in this region where u is close to v_i. It is in general difficult to develop an accurate estimate for this residual term. We can roughly estimate its value, however, by replacing $u-v_j$ with the smallest relative velocity that occurs between any pair of waves. Calling this difference δv, and proceeding under the conservative assumption that the contributions for different j add inphase, we then estimate

$$\overline{H}=\overline{\mathcal{E}}_\alpha^{(i)}+O\left[(N-1)\frac{\epsilon^2}{\delta v^2}\right], \tag{72}$$

and we require

$$(N-1)\frac{\epsilon^2}{\delta v^2}=o(\epsilon) \tag{73}$$

in order that $\mathcal{E}_\alpha^{(i)}$ be invariant through first order. In terms of the physical wave amplitudes $\varphi^{(i)}$, all of which are order ϵ, this condition can be expressed as

$$\frac{q_\alpha\varphi^{(i)}}{m_\alpha\delta v^2}\ll\frac{1}{N-1} \tag{74}$$

and must hold for all species α and wave indices $i=1,2,\ldots,N$. This condition reduces for $N=2$ to that of Eq. (67), but for larger N requires the wave amplitudes to decrease, for fixed δv^2, like $1/(N-1)$.

Now consider the $O(\epsilon^{3/2})$ residual contributions to the electric potential which come from the N resonant regions. When there are N waves, it is possible that the sum of these contributions from the various waves may be large compared to any one of these terms by itself. Again, supposing conservatively that these contributions add in-phase, the total residual contribution will be $O(N\epsilon^{3/2})$, and if this is to be negligible compared to the first-order terms it is necessary that $N\epsilon^{3/2}=o(\epsilon)$, or $N=o(\epsilon^{-1/2})$. This condition can be written in terms of the $O(\epsilon)$ physical wave amplitudes $\varphi^{(i)}$ as

$$\frac{q_\alpha\varphi^{(i)}}{m_\alpha\overline{v}_\alpha^2}\ll\frac{1}{N^2}, \tag{75}$$

where $\overline{v}_\alpha$ is the thermal velocity for species α. Like Eq. (74) above, Eq. (75) must hold for all α and all wave indices $i=1,2,\ldots,N$. Each of these conditions indicates that, if the superposition is to hold, the wave amplitudes must decrease as the number of component waves increases, the latter being a stronger condition on the $\varphi^{(i)}$ with respect to the number of waves, and the former giving an additional condition on the $\varphi^{(i)}$ with respect to the relative velocities between pairs of waves.

VII. APPLICATIONS

The superimposed solutions constructed here are relevant to the time-asymptotic plasma states observed in Vlasov-Poisson simulations of nonlinear Landay damping and of the growth and saturation of instabilities associated with the one- and two-sided bump-on-tail distributions. Based upon earlier work, Shoucri [21] has conjectured that in each of these cases the plasma approaches a BGK traveling wave, i.e., a self-consistent plasma state of the form $[f_\alpha=f_\alpha(x-vt),E=E(x-vt)]$, which implies the existence of a special frame of reference in which the distribution functions and field are time independent. While this idea is supported by some numerical evidence, especially in the case of the two-stream instability (which occurs in the presence of two counterpropagating beams of sufficient density and velocity), it cannot in general be correct. For if $(f_\alpha(x,u,t),\ E(x,t))$ is a solution to the Vlasov-Poisson system, then, since these equations are invariant under spatial reflection $(x,u)\rightarrow(-x,-u)$, $[f_\alpha(-x,-u,t),\ -E(-x,t)]$ is also a solution. By the uniqueness of solutions of the Vlasov-Poisson system, it then follows that a plasma described initially by refection-symmetric distribution functions $f_\alpha(x,u,0)=f_\alpha(-x,-u,0)$ will always remain in a reflection-symmetric state. Accordingly, the distribution functions and the electric field will satisfy

$$f_\alpha(x,u,t)=f_\alpha(-x,-u,t), \tag{76}$$

$$E(x,t)=-E(-x,t) \tag{77}$$

for all time. In other words, space-reflection symmetry is preserved by the dynamical evolution, and if present initially, must always be present. Equation (77) implies, in particular, that at $x=0$ we have $E(0,t)=-E(0,t)=0$, i.e., there is always a node in the electric field. The existence of such a node is clearly inconsistent with the BGK form $E=E(x-vt)$, which describes nodes propagating at velocity v. Indeed, the only BGK mode toward which a plasma in a reflection-symmetric initial state could evolve is one for which $v=0$, which is a nonpropagating final state. A simpler way to state the foregoing argument is to say that a reflection-symmetric plasma state cannot approach with time a single BGK wave of nonzero phase velocity, since to do so would break the reflection symmetry and thus violate the dynamics specified by the Vlasov-Poisson system.

In the light of this observation, the proposition that a plasma with a space-reflection-symmetric initial condition will asymptotically approach a single BGK wave is no longer tenable; the simplest logical generalization then is that the plasma instead approaches a superposition of such waves. Demeio and Zweifel [17] proposed this idea, emphasizing, however, that such superpositions are themselves not solutions of the Vlasov-Poisson equations. Nevertheless, the available numerical evidence concerning the phenomenon of nonlinear Landau damping, as well as those of growth and eventual saturation of the one- and two-sided bump-on-tail instabilities, suggests that the notion of superposed BGK waves is apparently necessary to the proper description of the observed time-asymptotic states in each of these cases. The results of the most recent and perhaps most accurate simulations of the Vlasov-Poisson system with periodic boundary conditions have been reported in Demeio and Zweifel [17], and similar results have also been obtained by Klimas [22]. In all the cases of interest here, the initial state of the plasma is given, in dimensionless units, by

$$f_e(x,u,0)=[1-kE_0\cos(kx)]F_0(u)\ , \tag{78}$$

$$E(x,0)=E_0\sin(kx)\ , \tag{79}$$

and is therefore reflection symmetric if and only if $F_0(u)$ is an even function of u.

The phenomenon of nonlinear Landau damping of an electrostatic wave is observed by taking $F_0(u)$ to be the Maxwellian distribution $F_0(u)=(2\pi)^{-1/2}\exp(-u^2/2)$ which, being single humped, is linearly stable. Demeio and Zweifel chose k in order to stimulate the longest wavelength mode for the system, which in the case of the Maxwellian is also the most weakly damped, and used $\epsilon=0.1$, which is sufficiently large for particle trapping to predominate before wave damping is complete. Their results show that, while a small quantity of energy leaks into other modes, the main electric field mode has much larger amplitude than any other throughout the entire process. After initially damping in agreement with the linear theory and then exhibiting the damped amplitude oscillations predicted by O'Neil [23], the field finally settles into a standing wave pattern, which indicates a superposition of two counterpropagating waves of equal amplitude and speed. In agreement with the symmetry property of the initial condition, the nodes of the field do not propagate; thus the asymptotic state is clearly not a single traveling BGK wave. The simulation also shows that the electron distribution function forms two phase space vortices centered at velocities $\pm v_p$, where v_p is the phase velocity corresponding to wave number k as calculated from the Landau dispersion relation to the linear theory. Since these waves are weakly damped, $v_p=\omega/k$ is also given by the Vlasov dispersion relation. It seems clear that such vortices correspond to particles that have become trapped in the electric potentials of the two counterpropagating waves. In fact, all evidence suggests strongly that the asymptotic state is well described by two superimposed small amplitude undamped BGK waves which propagate in opposite directions with equal speeds and amplitudes. The simulation was followed for a long time without any significant further change apparent in the state of the plasma.

Similar behavior is observed in the growth and saturation of linear instabilities. If a beam of sufficient velocity and density is added to the Maxwellian, then the distribution becomes linearly unstable for a certain range of wave numbers. This is the so-called one-sided bump-on-tail distribution, typically described by a distribution function of the form

$$F_0(u)=\frac{n_p}{\sqrt{2\pi}}e^{-u^2/2}+\frac{n_b}{\sqrt{2\pi}}e^{-(u-v_0)^2/v_t^2}\ , \tag{80}$$

where v_0 is the translational velocity of the beam and n_p, n_b, and v_t satisfy the normalization condition $n_p+n_bv_t=1$, where n_p and n_b are the host plasma and beam number densities, and v_t is the beam thermal velocity in units of the host plasma thermal velocity. Since $F_0(u)\neq F_0(-u)$ in this case, the initial state is not reflection symmetric. Again stimulating the longest wavelength mode, which is the only unstable mode, Demeio and Zweifel find that the field initially grows as expected from linear theory, exhibits decaying amplitude oscillations, and then settles into a steady state. In this case the nodes of the field do propagate, so that interpretation in terms of a single BGK wave is perhaps possible. However, the electron distribution function, in addition to exhibiting a vortex at the phase velocity of the unstable mode, also develops another much smaller vortex at the phase velocity of the most weakly damped mode (according to the linear theory). The existence of this second vortex rules out interpretation in terms of a single BGK mode. Instead, it appears again that the asymptotic state can be most easily described by the superposition of two BGK waves, although this time the wave amplitudes are unequal as evidenced by the different sized vortices.

The two-sided bump-on-tail distribution is identical to the one-sided bump-on-tail except that another counterpropagating beam is added so as to restore the symmetry $F_0(u)=-F_0(u)$. The simulation in this case shows behavior very similar to the one-sided distribution, with a period of linear growth followed by amplitude oscillations and eventual saturation, except that the final state is a standing wave with nonpropagating nodes, and is well described by a superposition of BGK waves of equal amplitude and speed propagating in opposite directions. Again this agrees with the implications of the symmetry argument presented above.

We have to this point overlooked a certain interpretive difficulty. The Vlasov-Poisson model suffers from an inconsistency that is caused by the development with time of structure in the distribution function on ever finer scales. This is the so-called "filamentation" problem which is a significant obstacle to the implementation of accurate long-time numerical simulations. Filamentation is a real dynamical phenomenon, not merely a numerical artifact, and reflects the underlying infinite-dimensional Hamiltonian structure of the Vlasov-Maxwell system [24]. In fact, the existence of this Hamiltonian structure raises an important question—what precisely do we mean when we speak, as we have, of the plasma approaching an asymptotic state? For Hamiltonian systems do not have attractors and therefore cannot, strictly speaking, have asymptotically stable final states.

The answer to this question results from carefully distinguishing what is physics from what is mathematics. Filamentation reflects, after a certain period of evolution, an unfortunate and inconsistent property of the otherwise very useful Vlasov-Maxwell model for collisionless plasmas. For the fundamental assumption of this model is that smoothly varying distribution functions are sufficient to accurately describe such a plasma. If so, then it is certainly inconsistent for the dynamics to introduce features which vary on scales shorter than the shortest scale over which f_α has meaning, a distance that typically is of the order of the Debye length $\lambda_D=(kT_e/4\pi ne^2)^{1/2}$. In comparing the time-developing state of a Vlasov-Poisson solution to a proposed smoothly varying distribution function it is appropriate, therefore, to ignore structure that develops on scales smaller than the resolution of the model. Such fine scale structure, which distinguishes among a very large number of microscopic states which correspond, however, to a single macroscopic physical state, can conveniently be removed by interpreting the distribution function in a coarse-grained sense, in which f_e is averaged over regions that are not too small relative to the Debye length λ_D. In fact, since any measuring device has a small but finite resolution that averages details on finer scales, coarse graining corresponds closely to the measurement process by which distribution functions would be obtained, at least in principle, from a physical plasma. In our preceding discussion of the relevant numerical studies of the Vlasov-Poisson system we have ignored the fine scale structure in two ways, first by focusing on the electric field, which does not suffer from the filamentation problem, and second by ignoring the small scale variation of the distribution function that occurs within the phase space vortices that have formed. When this fine scale structure is ignored, the evidence is clearly very strong that the plasma in each of the important cases cited does indeed approach (in a coarse-grained sense) an asymptotic state that is well described by the superposition of small-amplitude BGK waves—although the exact, fine-grained solution of the Vlasov-Poisson equations for the distribution functions may continue to evolve, becoming ever more phase mixed in the trapped particle regions.

VIII. SUMMARY AND CONCLUSIONS

It is generally accepted that the description offered by the linear theory is incorrect for a plasma that is sufficiently far from equilibrium. What is not widely known, however, is that the linear theory does not give an adequate description even for plasmas that are arbitrarily close to equilibrium. From a mathematical point of view, the validity of the linear approximation rests upon the condition

$$|\partial h_\alpha/\partial u| \ll |dF_\alpha/du| , \tag{81}$$

where $h_\alpha=f_\alpha-F_\alpha$. But, as we have seen, this condition is not an automatic consequence of the smallness of either h_α or $\varphi(\xi)$. In fact, there exist nonlinear traveling wave solutions of arbitrarily small amplitude that do not exhibit damping or growth, even when the linear theory suggests that they should. The distinctive feature of these waves, as opposed to those described by the linear theory, is that some of the plasma particles are trapped within the potential wells formed by the electric potential $\varphi(\xi)$. A close analysis shows, in fact, that the linear theory has no implications concerning the properties of these waves since, due to particle trapping, the distribution functions $f_\alpha=F_\alpha+h_\alpha$ must necessarily satisfy $(\partial f_\alpha/\partial u)|_{u=v}=0$, or $(\partial h_\alpha/\partial u)|_{u=v}=-(dF_\alpha/du)|_{u=v}$, even as the wave amplitude approaches zero. Thus, the condition given by Eq. (81) for the applicability of the linear approximation is clearly violated, and indeed there are waves of arbitrarily small amplitude to which the linear theory does not apply.

In this paper we have explored the interactions that occur between small-amplitude spatially periodic BGK waves when the relative velocity between the waves is not too small. Our study of this problem was motivated by a desire to explain the results of recent large scale numerical simulations of the Vlasov-Poisson system, and to understand in particular the asymptotic plasma states observed in the evolution of several fundamental nonlinear plasma phenomena, including nonlinear Landau damping and the growth of various linear instabilities. Due to the essentially nonlinear phenomenon of particle trapping, which remains important at any wave amplitude, a linear superposition of two spatially periodic BGK waves does not yield a new solution, even in the limit of zero wave amplitude, as was demonstrated in Sec. III. However, large scale numerical simulations performed by Demeio and Zweifel show final states that contain superpositions (in some sense) of two plasma waves, where the total electric field is, to a very good approximation, a simple linear superposition of two traveling electrostatic waves. Moreover, each of the waves maintains constant amplitude and, as indicated in the contours of the distribution function, contains trapped particles. These results strongly suggest that there perhaps exists a nonlinear superposition principle for small-amplitude BGK waves. Here we have developed this nonlinear superposition principle by explicitly constructing the distribution functions for a self-consistent superimposed two-wave state.

Our approach was to first understand the appropriate particle dynamics, and only then to attack the self-consistent plasma problem. By focusing on the nonintegrable Hamiltonian system for the motion of a charged particle in the field of two spatially periodic electrostatic waves, we were able to develop through perturbation methods two first-order invariants, generalizations of the single-particle energy, which capture the gross features of the dynamics, including the primary resonance regions corresponding to each of the two waves. We then used these invariants to construct smooth distribution functions for a two-wave state, and demonstrated that these functions satisfied the Vlasov equation uniformly through first order in the wave amplitudes and also generated, through the charge and current densities which enter into the inhomogeneous Maxwell equations, the correct self-consistent field, again through first order. The result was also generalized to the case of N waves for N not too large. The self-consistent multiple-wave solution shows that small-amplitude spatially periodic BGK waves, the velocities of which are not too close, do satisfy a nonlinear superposition principle in which the fields superimpose linearly while the distribution functions combine according to a somewhat more complicated but explicitly derived rule. This principle explains, in part, the asymptotic plasma states observed in the large scale numerical simulations mentioned above.

ACKNOWLEDGMENT

This research was supported by NASA under Grants Nos. NGT-53643 and NGW-1669.

---

[1] A. Vlasov, J. Phys. **9**, 25 (1945).
[2] L. Landau, J. Phys. **10**, 25 (1946).
[3] J. D. Jackson, J. Nucl. En. C **1**, 171 (1960).
[4] J. H. Malmberg and C. B. Wharton, Phys. Rev. Lett. **13**, 184 (1964).
[5] J. H. Malmberg, C. B. Wharton, and W. E. Drummond, in *Plasma Physics and Controlled Nuclear Fusion, Vol. 1* (International Atomic Energy Agency, Vienna, 1965).
[6] J. Dawson, Phys. Fluids **4**, 869 (1961).
[7] T. J. Dolan, *Fusion Research* (Pergamon, New York, 1982).
[8] Y. L. Al'pert, *Space Plasma* (Cambridge University Press, Cambridge, 1990), Vols. I and II.
[9] N. G. Van Kampen, Physica **21**, 949 (1955).
[10] K. M. Case, Ann. Phys. **7**, 349 (1959).
[11] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).
[12] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).
[13] J. P. Holloway and J. J. Dorning, Phys. Rev. A **44**, 3856 (1991). See also J. P. Holloway, Ph.D. dissertation, University of Virginia, 1989.
[14] M. Buchanan and J. J. Dorning, Phys. Lett. A **179**, 306 (1993).
[15] M. Buchanan, Ph.D. dissertation, University of Virginia, 1993.
[16] M. Buchanan and J. J. Dorning, Phys. Rev. Lett. **70**, 3732 (1993).
[17] L. Demeio and P. F. Zweifel, Phys. Fluids **2**, 1252 (1990). See also L. Demeio, Ph.D. dissertation, Virginia Polytechnic Institute and State University, 1989.
[18] H. Schamel, J. Plasma Phys. **7**, 1 (1972).
[19] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
[20] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA 1950).
[21] M. M. Shoucri, Phys. Fluids **22**, 2038 (1979).
[22] A. J. Klimas (private communication).
[23] T. O'Neil, Phys. Fluids **8**, 2255 (1965).
[24] P. J. Morrison, Phys. Lett. **80A**, 383 (1980).